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OPTIMAL DESIGN OF STRUCTURES OF COMPOSITE MATERIALS

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Abstract—Optimal design criteria for structures of composite materials are discussed assuming that the phases are either perfectly plastic or perfectly elastic and the design is aimed at maximizing limit load or minimizing static compliance of the structure. Both internal and external reinforcement are considered. An example of a circular plate is presented in order to illustrate the general criteria.

1. INTRODUCTION

PROBLEMS of optimization of uniform structures have been formulated for various design criteria [1–7]. For elastic structures, the optimal design for given material volume may correspond to minimum static or dynamic compliance measured as the work of external forces on induced displacements. For perfectly-plastic structures, the condition of maximum limit load is usually assumed in considering the optimization problem. In the case of free vibrations, the optimal design is sought that corresponds to maximum fundamental frequency, whereas in the case of buckling, the maximum buckling load is to be attained for the optimal solution. If no constraints are imposed on the design, the sufficient optimality criteria have a similar form for these cases: a certain function G, being the specific elastic energy or the specific power of plastic dissipation in the static case, and the difference of amplitudes of specific elastic and kinetic energies in the case of harmonic vibrations, should be constant on a traction-free boundary subjected to modification and should decrease in the direction of the exterior normal to that boundary. To obtain more practicable designs, some geometric constraints are often introduced; for instance, the free boundary is required to lie in some prescribed region or beyond some forbidden region. The modification of the design criteria in presence of geometric constraints has been discussed in [13].

In the present paper, we shall derive the optimality criteria for structures composed of several materials. In particular, the analysis may pertain to fibre-reinforced materials where thin fibres of high strength reinforce a weaker matrix, or to laminated materials where thin layers of reinforcement are introduced in order to augment stiffness or load carrying capacity of a structural element. The problem of optimization of reinforced concrete plates or shells has already been discussed in numerous papers [8–12]. It has been assumed that both reinforcement and concrete are perfectly plastic materials and tensile stresses are carried by reinforcement rods whereas the concrete carries compressive stresses. The problem of optimization of elastic, fibre-reinforced plates was considered in [15]. Here, we shall analyse an arbitrary composition of several materials without imposing in advance a form of particular phases. For given volume of phases, the optimal design should correspond to maximum limit load or to minimum elastic compliance, for perfectly plastic or elastic materials, respectively. In Section 2, we shall discuss the case of a two phase material with internal reinforcement. In Section 3, the case of external reinforcement will be considered and generalizations will be briefly discussed in Section 5. In Section 4, two examples will be presented of a plate with interior and exterior reinforcement.

2. OPTIMALITY CRITERIA FOR INTERIOR REINFORCEMENT

Let us consider first some typical cases of optimization of two-phase materials assuming that these are either perfectly plastic or perfectly elastic. The results can be generalized to other design criteria or to a greater number of phases.

2.1. *Criterion of the maximum load carrying capacity*

Figure 1(a) presents schematically a body loaded on the portion S_T of its boundary S, supported on the boundary S_u° so that the work of support reactions $\mathbf{T}_p$ on induced displacements $\mathbf{u}_p$ equals zero, $\int \mathbf{T}_p \cdot \mathbf{u}_p \, dS_u^\circ = 0$, (the dot between two symbols will denote the inner product of respective tensors or vectors), and with the free boundary S°. It is assumed that all portions S_T, S_u°, S° are prescribed and the boundary surface $S = S_T \cup S_u^\circ \cup S^\circ$ embraces the region of fixed volume V. Let this volume be occupied by two materials whose volumes V_r and V_m can vary. The material of volume V_r is of greater strength or stiffness and will be called the reinforcement whereas the weaker material of volume V_m will be called the matrix. Assuming now that both materials are perfectly plastic, the problem of optimization can be formulated as follows: for the specified boundary S and volume V, the given volume V_r of the reinforcement should be located in an optimal manner within V so that the limit load of the whole body should attain the maximum. The reinforcement can lie totally in the interior of the body or coincide on some portion AB with its boundary S. The modification of the form of reinforcement can thus be performed by altering its interior surface S_i whereas the portion AB coinciding with S can be regarded as fixed.

Consider the rigid, perfectly plastic structure in the limit state. Denote by $D_r(\dot{\boldsymbol{\varepsilon}}) = \boldsymbol{\sigma}_r \cdot \dot{\boldsymbol{\varepsilon}}$ and $D_m(\dot{\boldsymbol{\varepsilon}}) = \boldsymbol{\sigma}_m \cdot \dot{\boldsymbol{\varepsilon}}$ the specific powers of dissipation (per unit volume) of the reinforcement and the matrix. Since the reinforcement is stronger than the matrix, for any plastic strain rate $\dot{\boldsymbol{\varepsilon}}$ there is

$$D_r(\dot{\boldsymbol{\varepsilon}}) \geq D_m(\dot{\boldsymbol{\varepsilon}}). \tag{2.1}$$

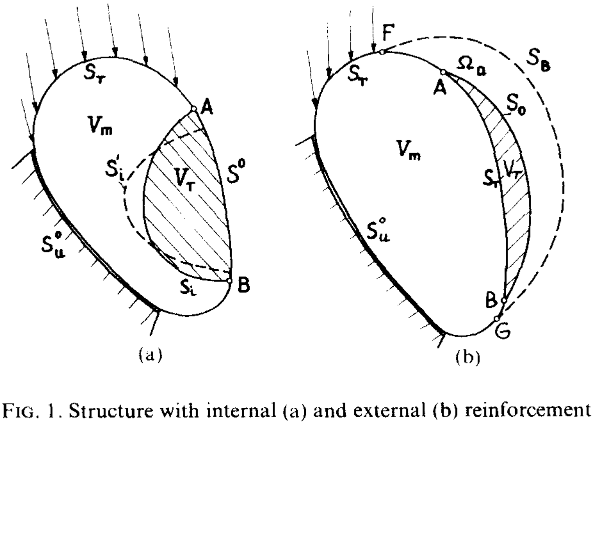

FIG. 1. Structure with internal (a) and external (b) reinforcement.

On the boundary S_i between the two phases the stress and the strain rate can vary discontinuously; however, the velocity vector $\mathbf{v}$ and the interaction force $\mathbf{R}$ are continuous functions at the transition between these phases. We can thus write

$$[R_i] = [\sigma_{ij}\nu_j] = 0, \qquad \left[\frac{\partial v_i}{\partial x_j}\right] = a_i\nu_j, \tag{2.2}$$

where square brackets denote jumps of enclosed quantities; ν_i denotes the unit vector normal to S_i and directed in the interior of V_m, and a_i is the discontinuity vector. The first equality (2.2) means that the interaction force on S_i is continuous and the second relation expresses the jump-like variation of the gradient $\partial v_i/\partial x_j$ at the transition from V_r to V_m. In view of (2.2) we see that the strain rates of elements tangential to S_i are continuous on S_i. Thus stress and strain rate discontinuities satisfy the equality

$$[\sigma_{ij}][\varepsilon_{ij}] = [\sigma_{ij}]\left[\frac{\partial v_i}{\partial x_j}\right] = [\sigma_{ij}]a_i\nu_j = [\sigma_{ij}\nu_j]a_i = 0. \tag{2.3}$$

The analysis of the stress state at the interface between two materials of different yield limits can be found in [14].

Denote by σ_r, σ_m the stress states within the two phases in the limit state. We can write

$$\int \boldsymbol{\sigma}_m \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m + \int \boldsymbol{\sigma}_r \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_r = \int \mathbf{T} \,.\, \mathbf{v}\, \mathrm{d}S_T \tag{2.4}$$

where $\mathbf{v}$ and $\dot{\boldsymbol{\varepsilon}}$ denote the velocity and strain rate fields defined throughout the region of volume V. Suppose now that the boundary S_i has been altered and denote the new boundary by S_i'. The stress state within the phases is now $\boldsymbol{\sigma}_r'$ and $\boldsymbol{\sigma}_m'$. This state satisfies the equilibrium equations, boundary conditions on S_T for the surface tractions $\lambda\mathbf{T}$ where λ is a positive multiplier. We can thus write the principle of virtual work for the stress state $\boldsymbol{\sigma}_r'$, $\boldsymbol{\sigma}_m'$ and the velocity field $\mathbf{v}$

$$\int \boldsymbol{\sigma}_m' \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m' + \int \boldsymbol{\sigma}_r' \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_r' = \lambda \int \mathbf{T} \,.\, \mathbf{v}\, \mathrm{d}S_T. \tag{2.5}$$

Since $V_r' = V_r + \Delta V$, $V_m' = V_m - \Delta V$ where V_r' and V_m' denote the volumes of the two phases after modification of the boundary S_i, subtracting (2.4) and (2.5), we obtain

$$\int (\boldsymbol{\sigma}_m' - \boldsymbol{\sigma}_m) \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m' + \int (\boldsymbol{\sigma}_r' - \boldsymbol{\sigma}_r) \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_r' + \\ + \int [D_r(\dot{\boldsymbol{\varepsilon}}) - D_m(\dot{\boldsymbol{\varepsilon}})]\, \mathrm{d}(\Delta V) = (\lambda - 1) \int \mathbf{T} \,.\, \mathbf{v}\, \mathrm{d}S_T, \tag{2.6}$$

where the stresses $\boldsymbol{\sigma}_r$ and $\boldsymbol{\sigma}_m$ are related to $\dot{\boldsymbol{\varepsilon}}$ by the associated flow law. Thus, writing

$$\int \boldsymbol{\sigma}_r \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_r = \int \boldsymbol{\sigma}_r \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_r' - \int \boldsymbol{\sigma}_r \,.\, \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}(\Delta V) \tag{2.7}$$

it is understood that within ΔV the stress state $\boldsymbol{\sigma}_r$ is related by the flow law to the plastic strain rate $\dot{\boldsymbol{\varepsilon}}$.

Starting from equation (2.6), the necessary and sufficient conditions of optimal design can be derived. We are looking for such a form of V_r and its situation within the region V which corresponds to a maximum of the load carrying capacity.

Denote $G = D_r(\dot{\varepsilon}) - D_m(\dot{\varepsilon})$ and by G_i^r the value of G on S_i. Since G may change discontinuously when passing from the interior of V_r to V_m, G_i^r will denote the value of G at boundary points of the region V_r, lying on S_i. Assume that S_i is so chosen that G_i^r is constant. Writing $G = G_i^r + G_\Delta$ and using the equality

$$\int G_i^r \, d(\Delta V) = 0, \tag{2.8}$$

equation (2.6) will take the form

$$(\lambda - 1)\int \mathbf{T} \cdot \mathbf{v} \, dS_T = \int G_\Delta \, d(\Delta V) + \int (\boldsymbol{\sigma}_m' - \boldsymbol{\sigma}_m) \cdot \dot{\boldsymbol{\varepsilon}} \, dV_m' + \int (\boldsymbol{\sigma}_r' - \boldsymbol{\sigma}_r) \cdot \dot{\boldsymbol{\varepsilon}} \, dV_r. \tag{2.9}$$

If $G \geq G_i^r$ in the interior of V_r, and $G \leq G_i^r$ within V_m, then $G_\Delta \, d(\Delta V) \leq 0$; the two remaining integrands on the right-hand side of (2.9) are non-negative in view of the principle of maximum plastic work [16]. Thus $\lambda \leq 1$ and any modification of S_i cannot lead to increase of the limit load. The set of sufficient conditions for absolute maximum of the limit load can thus be expressed as follows

$$G_i^r = D_r(\dot{\varepsilon}) - D_m(\dot{\varepsilon}) = \text{const.} \qquad \text{on } S_i, \tag{2.10a}$$

$$G(\mathbf{x}) \geq G_i^r \qquad \text{for } \mathbf{x} \in V_r, \qquad G(\mathbf{x}) \leq G_i^r \qquad \text{for } \mathbf{x} \in V_m. \tag{2.10b}$$

Identical conditions could be derived if we assumed the limit load to be constant and sought for the optimal form of the region V_r corresponding to the minimum of reinforcement volume. In fact, setting in (2.6) $\lambda = 1$ and using (2.10a) we obtain

$$G_i^r \Delta V = \int (\boldsymbol{\sigma}_m - \boldsymbol{\sigma}_m') \cdot \dot{\boldsymbol{\varepsilon}} \, dV_m' + \int (\boldsymbol{\sigma}_r - \boldsymbol{\sigma}_r') \cdot \boldsymbol{\varepsilon} \, dV_r' - \int G_\Delta \, d(\Delta V). \tag{2.11}$$

Thus $\Delta V \geq 0$ provided the conditions (2.10) are fulfilled. The necessity of the condition (2.10a) for a local extremum of the limit load and the condition (2.11b) for an absolute maximum can also be demonstrated by departing from equation (2.6).

2.2 *Criterion of the minimum elastic compliance*

Assume now that the two materials are linearly elastic, satisfying the generalized Hooke's law. The optimal shape of V_r will be sought that corresponds to the minimum of elastic compliance measured as the work of surface tractions on induced displacements

$$I = \int \mathbf{T} \cdot \mathbf{u} \, dS_T. \tag{2.12}$$

Both the reinforcement and the matrix are characterized by a positive-definite specific elastic energy $E = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$. For any $\boldsymbol{\varepsilon}$, we have

$$E_r(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\boldsymbol{\sigma}_r \cdot \boldsymbol{\varepsilon} \geq E_m(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\boldsymbol{\sigma}_m \cdot \boldsymbol{\varepsilon}, \tag{2.13}$$

where $\boldsymbol{\sigma}_r$ and $\boldsymbol{\sigma}_m$ denote stresses in particular phases, related by the Hooke's law to the strain $\boldsymbol{\varepsilon}$.

Similarly as previously, assume that $\boldsymbol{\sigma}_r, \boldsymbol{\sigma}_m, \boldsymbol{\varepsilon}, \mathbf{u}$ denote stress, strain and displacement fields within the regions V_r and V_m; $\boldsymbol{\sigma}'_r, \boldsymbol{\sigma}'_m, \boldsymbol{\varepsilon}', \mathbf{u}'$ are the corresponding fields after modification of S_i. The new regions occupied by the reinforcement and the matrix are denoted by V'_r and V'_m. According to the principle of virtual work, we can write

$$\int \mathbf{T} \,.\, \mathbf{u} \,\mathrm{d}S_T = \int \boldsymbol{\sigma}_r \,.\, \boldsymbol{\varepsilon} \,\mathrm{d}V_r + \int \boldsymbol{\sigma}_m \,.\, \boldsymbol{\varepsilon} \,\mathrm{d}V_m = \int \boldsymbol{\sigma}'_r \,.\, \boldsymbol{\varepsilon} \,\mathrm{d}V'_r + \int \boldsymbol{\sigma}'_m \,.\, \boldsymbol{\varepsilon} \,\mathrm{d}V'_m,$$

$$\int \mathbf{T} \,.\, \mathbf{u}' \,\mathrm{d}S_T = \int \boldsymbol{\sigma}'_r \,.\, \boldsymbol{\varepsilon}' \,\mathrm{d}V'_r + \int \boldsymbol{\sigma}'_m \,.\, \boldsymbol{\varepsilon}' \,\mathrm{d}V'_m. \tag{2.14}$$

Using (2.14), the variation of elastic compliance due to modification of the region V_r can be expressed as follows

$$I' - I = \int \mathbf{T} \,.\, (\mathbf{u}' - \mathbf{u}) \,\mathrm{d}S_T = \int \Delta\boldsymbol{\sigma}_r \,.\, \Delta\boldsymbol{\varepsilon} \,\mathrm{d}V'_r + \int \Delta\boldsymbol{\sigma}_m \,.\, \Delta\boldsymbol{\varepsilon} \,\mathrm{d}V'_m - \int G_\Delta \,\mathrm{d}(\Delta V), \tag{2.15}$$

where the differences $\Delta\boldsymbol{\sigma}_r = \boldsymbol{\sigma}'_r - \boldsymbol{\sigma}_r$ and $\Delta\boldsymbol{\sigma}_m = \boldsymbol{\sigma}'_m - \boldsymbol{\sigma}_m$ are related by the Hooke's law to the strain difference $\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}$; G similarly as previously, denotes the doubled difference of specific energies, $G = 2E_r(\boldsymbol{\varepsilon}) - 2E_m(\boldsymbol{\varepsilon})$.

Let G_i^r denote the value of G at points of V_r lying on S_i. If G_i^r is constant on S_i, then setting $G = G_i^r + G_\Delta$, equation (2.15) can be presented in the form

$$I' - I = \int 2E_r(\Delta\boldsymbol{\varepsilon}) \,\mathrm{d}V'_r + \int 2E_m(\Delta\boldsymbol{\varepsilon}) \,\mathrm{d}V'_m - \int G_\Delta \,\mathrm{d}(\Delta V). \tag{2.16}$$

If $G(\mathbf{x}) \geq G_i^r$ for $\mathbf{x} \in V_r$ and $G(\mathbf{x}) \leq G_i^r$ for $\mathbf{x} \in V_m$, then from (2.16) it follows that $I' \geq I$ since all terms on the right hand side of (2.16) are non-negative. Thus the conditions analogous to (2.10) assure the absolute minimum of the elastic compliance.

3. OPTIMALITY CRITERIA FOR EXTERIOR REINFORCEMENT

Now, let us consider the case when the reinforcement is located in the exterior of the body of volume V_m and of specified shape; on the portion S_u° the body is supported and is loaded on the portion S_T of its boundary by the prescribed surface tractions $\mathbf{T}$. Assume that the reinforcement of given volume V_r is attached to the free boundary S_r of the body, see Fig. 1(b). The connection of the two phases is assumed to be perfect, that is there is a discontinuous change from the properties of the matrix to those of the reinforcement when passing the interface S_r. The modification of the reinforcement shape can be performed by modifying its traction-free surface S_0 which should lie within some admissible region Ω_a bounded by the surface S_B.

We shall confine ourselves to considering the limit state of the structure and derive the criteria of optimal shape of the boundary S_0, which corresponds to the greatest limit load for constant volume of the reinforcement.

Let $\boldsymbol{\sigma}_m, \boldsymbol{\sigma}_r, \dot{\boldsymbol{\varepsilon}}, \mathbf{v}$ denote the stress, strain-rate and velocity fields within the two phases in the limit state. Denote by $\boldsymbol{\sigma}'_r, \boldsymbol{\sigma}'_m$ the stress state upon changing the boundary S_0 to S'_0; this state satisfies the internal equilibrium equations and boundary conditions on S'_0

and S_T for the surface tractions $\lambda\mathbf{T}$. We can thus write

$$\int \mathbf{T} \cdot \mathbf{v}\, \mathrm{d}S_T = \int \boldsymbol{\sigma}_m \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m + \int \boldsymbol{\sigma}_r \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_r \tag{3.1}$$

$$(\lambda - 1)\int \mathbf{T} \cdot \mathbf{v}\, \mathrm{d}S_T = \int \boldsymbol{\sigma}'_m \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m + \int \boldsymbol{\sigma}'_r \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V'_r \tag{3.2}$$

In writing (3.2), it has been assumed that the velocity field can be continued beyond S_0 and is defined within the whole admissible region Ω_a. Subtracting (3.1) from (3.2) we have

$$(\lambda - 1)\int \mathbf{T} \cdot \mathbf{v}\, \mathrm{d}S_T = \int (\boldsymbol{\sigma}'_m - \boldsymbol{\sigma}_m) \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m + \int (\boldsymbol{\sigma}'_r - \boldsymbol{\sigma}_r) \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V'_r - \int \boldsymbol{\sigma}_r \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}(\Delta V). \tag{3.3}$$

Denoting $G = \boldsymbol{\sigma}_r \cdot \dot{\boldsymbol{\varepsilon}} = G(\dot{\boldsymbol{\varepsilon}})$ and assuming that $G = G_0 = \text{const.}$ on S_0, equation (3.3), upon using (2.8), can be presented in the form

$$(\lambda - 1)\int \mathbf{T} \cdot \mathbf{v}\, \mathrm{d}S_T = \int (\boldsymbol{\sigma}'_m - \boldsymbol{\sigma}_m) \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V_m + \int (\boldsymbol{\sigma}'_r - \boldsymbol{\sigma}_r) \cdot \dot{\boldsymbol{\varepsilon}}\, \mathrm{d}V'_r - \int G_\Delta\, \mathrm{d}(\Delta V), \tag{3.4}$$

when, similarly as previously, $G = G_0 + G_\Delta$. Equation (3.4) is analogous to (2.9) and the criteria of absolute maximum of λ are identical to (2.10) provided G_i^r is replaced by G_0. Thus in the whole admissible region beyond S_0 there should be $G \leq G_0$ and in the interior of V_r the inequality $G \geq G_0$ should be satisfied. In particular, if the reinforcement does not lie over the whole boundary S_r but only on its portion AB the specific power of dissipation on portions AF and BG should not be greater than G_0.

The present formulation of the optimal reinforcement problem can also be regarded as the problem of *optimal adaptation* of a structure. Let the body of volume V_m be designed to carry some prescribed loads T_0. When the loading has changed from T_0 to $\mathbf{T}_1$ or some additional loads are superimposed, the structure should be optimally adapted to this new loading. This adaptation can be performed by adding a material in some parts of the basic structure which are weakest for the new loading; this is equivalent to considering exterior reinforcement and the criteria of optimal adaptation are the same as those of optimal exterior reinforcement.

For elastic structures, the criteria of optimal reinforcement corresponding to a minimum of elastic compliance are identical. The specified elastic energy is now used as the function G; this should be constant on S_0 and decrease when passing from the interior of V_r into the admissible region Ω_a.

4. EXAMPLES

Consider a circular plate of radius a, thickness $2h$, simply supported, and uniformly loaded by the lateral load q. When the plate is made of a uniform material satisfying the Tresca yield condition, the limit load q_1 of the plate equals

$$q_1 a^2 = 6M_{01}, \tag{4.1}$$

where $M_{01} = \sigma_{01} h^2$ is the limit bending moment in uniaxial flexure and σ_{01} is the yield limit in tension. In order to enhance the limit load, the plate is internally reinforced by a perfectly plastic uniform material, obeying the Tresca yield condition with the yield limit

$\sigma_{02} > \sigma_{01}$. The limit load of the reinforced plate will thus lie within the interval

$$6M_{01} < qa^2 < 6M_{02}, \tag{4.2}$$

where $M_{02} = \sigma_{02}h^2$. We assume that the reinforcement is placed in the central region $0 \le r \le \rho$; since the plastic power of dissipation increases with the distance from the middle plane, the exterior boundary of the reinforcement should coincide with the lateral plate surfaces and the function $z = z(r)$ defines its interior boundary subject to variation, Fig. 2(a).

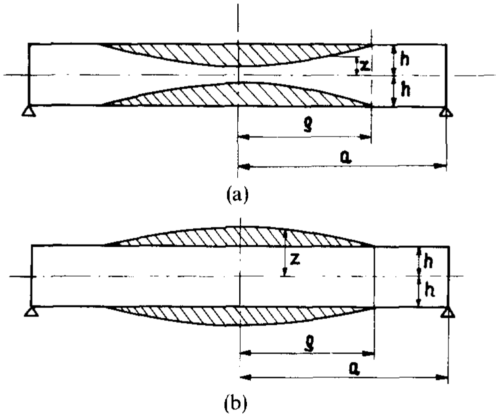

FIG. 2. Circular plate, simply supported: (a) interior, (b) exterior reinforcement.

4.1. Assume that for $0 \le r \le \rho$ the stress state is represented by a corner of the Tresca hexagon for which the radial and circumferential bending moments M_r and M_θ are equal, that is $M_r = M_\theta = M_0, \dot{\varkappa}_r \ge 0, \dot{\varkappa}_\theta \ge 0$, where

$$M_0 = \sigma_{02}(h^2 - z^2) = M_{02} - (\sigma_{02} - \sigma_{01})z^2, \tag{4.3}$$

and $\dot{\varkappa}_r$, $\dot{\varkappa}_\theta$ are the radial and circumferential rates of curvature. From the equilibrium equation

$$\frac{\mathrm{d}}{\mathrm{d}r}(rM_r) - M_\theta = -\tfrac{1}{2}qr^2, \tag{4.4}$$

and (4.3), we obtain for $0 \le r \le \rho$

$$M_r = -\tfrac{1}{4}qr^2 + C_1, \tag{4.5}$$

where C_1 is a constant. In the region $\rho \le r \le a$ we assume the stress state corresponding to the side of the Tresca hexagon for which $M_\theta = M_{01} = \sigma_{01}h^2$, $\dot{\varkappa}_r = 0$, $\dot{\varkappa}_\theta > 0$. From the equilibrium equation (4.4), we find

$$M_r = -M_0(a-r) + \tfrac{1}{6}q(a^3 - r^3). \tag{4.6}$$

Satisfying the continuity condition of M_r for $r = \rho$, equation (4.5) takes the form

$$M_r = \tfrac{1}{4}q(\rho^2 - r^2) - M_{01}\left(\frac{a}{\rho} - 1\right) + \tfrac{1}{6}q\left(\frac{a^3}{\rho} - \rho^2\right). \tag{4.7}$$

From (4.7) and (4.3) we can determine the function $z = z(r)$, namely

$$(\sigma_{02} - \sigma_{01})z^2 = M_{02} - \tfrac{1}{4}q(\rho^2 - r^2) + M_{01}\left(\frac{a}{\rho} - 1\right) - \tfrac{1}{6}q\left(\frac{a^3}{\rho} - \rho^2\right). \tag{4.8}$$

Since for $r = \rho$ there is $M_r = M_{01}$, from (4.6) we obtain

$$\frac{\rho}{a} = \left(1 - \frac{6M_{01}}{qa^2}\right)^{\frac{1}{2}}. \quad (4.9)$$

The above static field can be applied when $z(0) \geq 0$ which can be expressed by the condition

$$q\rho^2 \leq 4(M_{02} - M_{01}). \quad (4.10)$$

For higher values of q, the reinforcement will fill the entire region $z = \pm h$ within some interval $0 \leq r \leq \rho_1$. This case will not be discussed here.

Let us now consider the kinematic field. Since the two materials satisfy the Tresca yield condition, the optimality criterion for the corner of the Tresca hexagon for which $M_r = M_\theta$, takes the form

$$G_i = \sigma_r \dot{\varepsilon}_r + \sigma_\theta \dot{\varepsilon}_\theta = (\sigma_{02} - \sigma_{01})(\dot{\varepsilon}_r + \dot{\varepsilon}_\theta) = \text{const.} \quad (4.11)$$

on the surface $z = z(r)$. The condition (4.11) leads to the following differential equation

$$z\left(\frac{\mathrm{d}^2\dot{w}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\dot{w}}{\mathrm{d}r}\right) = -\alpha, \quad (4.12)$$

where α is a positive constant and $\dot{w}$ is the rate of plate deflection. Using (4.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\dot{w}}{\mathrm{d}r}\right) = -\frac{\alpha}{B}\frac{Br}{\sqrt{(A + Br^2)}}, \quad (4.13)$$

where

$$A = \left[M_{02} + M_{01}\left(\frac{a}{\rho} - 1\right) - \frac{1}{4}q\rho^2 - \frac{1}{6}q\left(\frac{a^3}{\rho} - \rho^2\right)\right](\sigma_{02} - \sigma_{01})^{-1}. \quad (4.14)$$

$$B = \frac{q}{4(\sigma_{02} - \sigma_{01})}.$$

In the region $\rho \leq r \leq a$ the stress state corresponds to the side of the Tresca hexagon for which $M_\theta = M_{01}$, and $\dot{\varkappa}_r = -\mathrm{d}^2\dot{w}/\mathrm{d}r^2 = 0$. Hence

$$\dot{w}_2 = D(a - r), \quad (4.15)$$

where D is constant. Integrating (4.13) and satisfying the continuity conditions of $\dot{w}$ and $\mathrm{d}\dot{w}/\mathrm{d}r$ for $r = \rho$ and the condition $\mathrm{d}\dot{w}/\mathrm{d}r = 0$ for $r = 0$, we find

$$0 \leq r \leq \rho: \quad \dot{w}_1 = \frac{\alpha}{B}\left[\left(\frac{\sqrt{(A + B\rho^2)}}{\rho} - \frac{\sqrt{A}}{\rho}\right)(a - \rho) - \sqrt{(A + Br^2)} + \sqrt{(A + B\rho^2)}\right.$$
$$\left. + \sqrt{\frac{A}{B}}\ln\frac{\sqrt{A} + \sqrt{(A + Br^2)}}{\sqrt{A} + \sqrt{(A + B\rho^2)}}\frac{\rho}{r} + \sqrt{A}\ln\frac{r}{\rho}\right], \quad (4.16)$$

$$\rho \leq r \leq a: \quad \dot{w}_2 = \frac{\alpha}{B}\left[\frac{\sqrt{(A + B\rho^2)}}{\rho} - \frac{\sqrt{A}}{\rho}\right](a - r). \quad (4.17)$$

It can be checked that within the whole plate there is $\dot{\varkappa}_r \geq 0$ and $\dot{\varkappa}_\theta \geq 0$; thus the kinematic field (4.17) satisfies the flaw laws for the Tresca yield condition. The power of dissipation is constant on the surface $z = z(r)$ and is smaller in the exterior of reinforcement than in its interior. Thus all optimality conditions (2.10) are satisfied and the solution (4.8) corresponds to the absolute maximum of the limit load for constant volume of the reinforcement.

4.2. Assume that the plate of constant thickness $2h$ is reinforced by adding the reinforcement on the lateral surfaces, Fig. 2(b). The shape of reinforcement is defined by the function $z = z(r)$. In the region $0 \leq r \leq \rho$ the limit bending moment is expressed as follows

$$M_0 = M_{01} + \sigma_{02}(z^2 - h^2) = M_{01} - M_{02} + \sigma_{02} z^2, \tag{4.18}$$

where $M_{01} = \sigma_{01} h^2$. The stress within the plate is determined by the formulae (4.6) and (4.7). The function $z = z(r)$ will be determined from (4.18) and (4.7)

$$\sigma_{02} \,.\, z^2 = M_{02} - M_{01}\frac{a}{\rho} + \frac{1}{4}q(\rho^2 - r^2) + \frac{1}{6}q\left(\frac{a^3}{\rho} - \rho^2\right), \tag{4.19}$$

and the value of ρ is found from (4.9). This solution is valid for arbitrary values of q, larger than the limit load of the non-reinforced plate, $qa^2 > 6M_{01}$.

The kinematic field is defined by (4.17), where

$$A\sigma_{02} = M_{02} - M_{01}\frac{a}{\rho} + \frac{1}{6}q\left(\frac{a^3}{\rho} - \rho^2\right), \qquad B = -\frac{q}{4\sigma_{02}}. \tag{4.20}$$

In the present case, the specific power of dissipation is constant on the free surface $z = z(r)$ of the reinforcement, $G = G_0 =$ const., and is smaller than G_0 on the remaining part of the lateral surface $z = \pm h$, $\rho \leq r \leq a$. However, the specific power of dissipation increases with the distance from the middle plane and hence it is larger outside of the free surface $z = z(r)$. Therefore the presented solution corresponds only to local extremum of the limit load and may not represent the optimal solution. Similarly as in the case of solid plates, the reinforcement could be better utilized in the form of discrete ribs arranged radially and circumferentially. The problem of correct formulation of optimal design problem for solid plates has been discussed in [21] and all conclusions there reached can be applied to the present case of exterior reinforcement.

5. GENERALIZATIONS

Let us now consider some generalizations of two fundamental cases discussed in Sections 2 and 3.

5.1. Assume that the internal reinforcement of the volume V_r is located within the matrix of volume V_m; however, both the free boundary S_0 of the body and the internal boundary S_i of the reinforcement are subjected to modifications which are independent of each other. The optimal form of S_0 and S_i should be determined for which the maximum load carrying capacity is attained.

Let $\boldsymbol{\sigma}_r$, $\boldsymbol{\sigma}_m$ and $\boldsymbol{\sigma}'_r$, $\boldsymbol{\sigma}'_m$ be the stress states corresponding to two shapes S_0, S'_0 and S_i, S'_i. Proceeding analogously as in Section 2, we can write

$$\int (\boldsymbol{\sigma}'_r - \boldsymbol{\sigma}_r) \,.\, \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}V_r + \int (\boldsymbol{\sigma}'_m - \boldsymbol{\sigma}_m) \,.\, \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}V'_m - \int \boldsymbol{\sigma}_r \,.\, \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}(\Delta V_r) + \int \boldsymbol{\sigma}_m \,.\, \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}(\Delta V_r) - \int \boldsymbol{\sigma}_m \,.\, \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}(\Delta V_m)$$

$$= (\lambda - 1) \int \mathbf{T} \,.\, \mathbf{v} \,\mathrm{d}S_T \tag{5.1}$$

where $\mathbf{v}$ and $\dot{\boldsymbol{\varepsilon}}$ are the velocity and strain rate fields for the body with the boundaries S_0 and S_i; the volumes after variation of S_0 and S_i are $V_r' = V_r + \Delta V_r$ and $V_m' = V_m - \Delta V_r + \Delta V_m$. Denote

$$G_0 = \boldsymbol{\sigma}_m \cdot \dot{\boldsymbol{\varepsilon}} = D_m(\dot{\boldsymbol{\varepsilon}}), \qquad G_i^r = D_r(\dot{\boldsymbol{\varepsilon}}) - D_m(\dot{\boldsymbol{\varepsilon}}). \tag{5.2}$$

From (5.1) it follows that when the functions G_0 and G_i^r are constant on S_0 and S_i, respectively, and decrease when passing in the exterior of V_m and V_r, the absolute maximum of the limit load is attained.

The problem is somewhat modified when the limit load is prescribed and the optimal form of S_0 and S_i is sought that corresponds to a minimum of total cost of materials. Introducing the cost function in the form

$$K = c_m V_m + c_r V_r, \tag{5.3}$$

where c_m and c_r are specific costs of the matrix and the reinforcement. From (5.1) the optimality conditions are obtained in the form

$$G_i^r = \frac{D_r(\dot{\boldsymbol{\varepsilon}}) - D_m(\dot{\boldsymbol{\varepsilon}})}{c_r} = \alpha \quad \text{on} \quad S_i, \qquad G_0 = \frac{D_m(\dot{\boldsymbol{\varepsilon}})}{c_m} = \alpha \quad \text{on} \quad S_0, \tag{5.4}$$

where α is constant. It should be noted that for the problem so formulated there can be solutions for which $V_m = 0$ or $V_r = 0$ when the conditions (5.4) cannot be satisfied simultaneously.

5.2. Consider a composition of n materials of different yield limits and of volumes V_{rk}, $k = 1, 2, \ldots, n$; it is assumed that the yield limits increase with k. An optimal system is obtained by placing one phase in the interior of the other so that the functions G_k^r should remain constant on the boundaries of adjacent phases and decrease when passing from a stronger to a weaker phase. On the exterior free boundary S_0 belonging to one of the phases, the function G should also be constant and decrease in the outside of S_0. Thus we have

$$\begin{aligned} G_k^r &= D_k(\dot{\boldsymbol{\varepsilon}}) - D_{k-1}(\dot{\boldsymbol{\varepsilon}}) = \text{const.} \quad \text{on} \quad S_{ik}, \qquad k = 2, 3, \ldots, n \\ G_l^0 &= D_l(\dot{\varepsilon}) = \text{const.} \quad \text{on} \quad S_0, \end{aligned}$$

where G_k^r denotes the value of G at points of the phase k lying on the boundary with the weaker phase $k-1$ and G_l^0 denotes the specific power of dissipation on the free boundary S_0.

These general statements coincide in particular cases with theorems on optimal shapes of multiply-connected cross sections of prismatic bars subjected to torsion [17–19]. Thus, for instance, according to theorems given in [18] and [19], from all multiply-connected cross sections of given area and given joint area of holes, a ring bounded by two concentric circles has the greatest torsional stiffness [18] or the greatest limit load [19]. Since all holes can be treated as a weaker material of zero yield limit, thus according to (5.5) these should be located in the vicinity of the bar axis and the dissipation power should be constant on the surface bounding the holes and should decrease toward the bar axis; the dissipation power should also be constant on the exterior boundary S_0 and these conditions are satisfied for an annular shape of the cross section. Only the condition of decreasing of $D(\dot{\varepsilon})$ in the exterior of S_0 is not satisfied. Similarly as from (2.9) and (2.16), an extremum of the limit load or the static compliance follows from (5.1) and from the analogous equation for an elastic structure. In fact, all terms on the right-hand side of (2.9) or (2.16) are infinitesimal quantities of the second order if infinitesimal variation of the free boundary is considered (for more detailed discussion, see [3] and [4]). The theorems given in [18] and [19] are stronger since they state that this extremum is a maximum.

6. CONCLUDING REMARKS

In deriving the optimality criteria for composites, we have not imposed any constraints on the reinforcement form; the optimal form should thus follow from the optimality criterion. For instance, if a sheet is subjected to uniform uniaxial tension, the given volume of reinforcement is utilized in the best way in the form of thin fibres arranged along the lines of tensile stresses. The optimal forms of reinforcement in plates are shown in Fig. 2 for the case of axial symmetry. When some constraints are imposed on the form of reinforcement, the optimality criteria should be modified similarly as the respective criteria for uniform structures with geometric constraints [4, 13]. For instance, when in the examples of Section 4 the reinforcement is introduced in the form of annuli of segmentwise varying thickness, the mean values of the specific power of dissipation or elastic energy should be the same for each segment. For fibre reinforced materials, we can obtain the criteria previously derived in [8–12], requiring the rate of elongation of fibres to be constant within the structure.

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Абстракт—Рассматриваются условия оптимального проектирования конструкции состоящих из несколько фаз. Допускается что фазы являются либо идеально упругими либо идеально пластичными. Рассматриваются два случая армирования: внутреннее и внешнее. В качестве илюстрации общей теорци дан пример круговой пластинки с внутренним и внешним армированием.